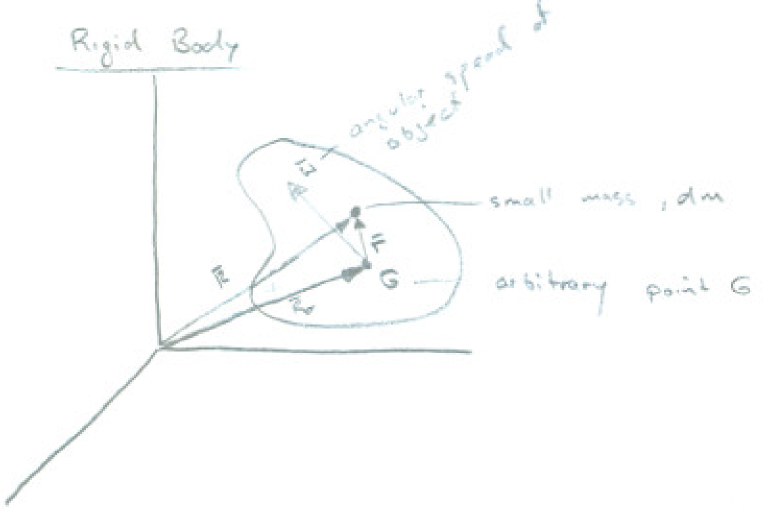


$$\Sigma \bar{F} = \frac{d\bar{p}}{dt} \text{ - momentum} \qquad \Sigma \bar{M} = \frac{d\bar{h}}{dt} \text{ - angular momentum}$$

$$\bar{R} = \bar{R}_G + \bar{r}$$

$\bar{v}$ = velocity of dm

$$= \frac{d\bar{R}_G}{dt} + \cancel{\frac{d\bar{r}}{dt}} + \bar{\omega} \times \bar{r}$$

$\to 0$

$\bar{r}$ is a fixed vector

$$d\bar{H}_0 = \bar{R} \times (dm \cdot \bar{v}) = (\bar{R}_G + \bar{r}) \times (\bar{v}_G + \bar{\omega} \times \bar{r})$$

$$= \bar{R}_G \times \bar{v}_G \cdot dm + R_G \times (\bar{\omega} \times \bar{r}) dm + \bar{r} \times \bar{v}_G dm + \bar{r} \times (\bar{\omega} \times \bar{r}) dm$$

$$\bar{H}_0 = \bar{R}_G \times \bar{v}_G \int_m dm + \bar{R}_G \times \bar{\omega} \times \int_m \bar{r} dm - \bar{v}_G \times \int_m \bar{r} dm + \int_m \bar{r} \times (\bar{\omega} \times \bar{r}) dm$$

if point G is the center of mass $\Rightarrow \int_m \bar{r} dm = 0$

also $\int_m dm = M$ - total mass of object

$$\bar{H}_0 = \bar{R}_G \times \bar{v}_G m + \underbrace{\int_m \bar{r} \times (\bar{\omega} \times \bar{r}) dm}_{\text{angular momentum about G}}$$

$$\int_m \underbrace{\bar{r} \times (\bar{\omega} \times \bar{r})}_{\text{mass distribution}} dm = \underbrace{\mathbb{I}}_{\text{inertial tensor}} \bar{\omega}$$

$$\bar{H}_0 = \bar{R}_G \times \bar{v}_G m + \underbrace{\bar{H}_G}_{\text{angular momentum about G}}$$

$$H_G = \int_m \bar{r} \times (\bar{\omega} \times \bar{r}) dm = \omega \int_m r^2 dm - \int_m \bar{r} (\bar{r} \cdot \bar{\omega}) dm$$

note: $\bar{A} \times (\bar{B} \times \bar{C}) = \bar{B}(\bar{A} \cdot \bar{C}) - \bar{C}(\bar{A} \cdot \bar{B})$

$\bar{r} \times (\bar{\omega} \times \bar{r}) = \omega \cdot (r^2) - \bar{r}(\bar{r} \cdot \bar{\omega})$

in cartesian coordinates: $\bar{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$

$\bar{r} = x \hat{i} + y \hat{j} + z \hat{k}$

$$\bar{r} \cdot (\bar{r} \cdot \bar{\omega}) = \bar{r}(x \omega_x + y \omega_y + z \omega_z)$$

expanding integrals

$$H_G = (\omega_x \hat{\imath} + \omega_y \hat{\jmath} + \omega_z \hat{k}) \int_m (x^2+y^2+z^2)\,dm - \int_m (x\omega_x + y\omega_y + z\omega_z)(x\hat{\imath} + y\hat{\jmath} + z\hat{k})\,dm$$

combining like terms

$$H_G = \int_m \left\{ \begin{aligned} &\left[(\cancel{x^2}+y^2+z^2)\omega_x - (\cancel{x^2\omega_x} + xy\omega_y + xz\omega_z)\right]\hat{\imath} + \\ &\left[(x^2+\cancel{y^2}+z^2)\omega_y - (yx\omega_x + \cancel{y^2\omega_y} + yz\omega_z)\right]\hat{\jmath} + \\ &\left[(x^2+y^2+\cancel{z^2})\omega_z - (zx\omega_x + zy\omega_y + \cancel{z^2\omega_z})\right]\hat{k} \end{aligned} \right\} dm$$

$$= \int_m \left\{ \begin{aligned} &\left[(y^2+z^2)\omega_x - (xy)\omega_y - (xz)\omega_z\right]\hat{\imath} + \\ &\left[-(xy)\,\omega_x + (x^2+z^2)\omega_y - (yz)\omega_z\right]\hat{\jmath} + \\ &\left[-(zx)\,\omega_x - (zy)\omega_y + (x^2+y^2)\omega_z\right]\hat{k} \end{aligned} \right\} dm$$

$$\underbrace{I_{xx} = \int_m (y^2+z^2)\,dm}_{} \qquad \underbrace{I_{xy} = -\int_m xy\,dm}_{}$$

moments of inertia
- represent mass distribution around an axis

products of inertia
- represent symmetry of mass distribution in the plane

$$H_G = \underbrace{\begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{xy} & I_{yy} & I_{yz} \\ I_{xz} & I_{yz} & I_{zz} \end{bmatrix}}_{\text{symmetric tensor } \mathbb{I}} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}$$

fix xyz in the rigid body $\therefore$ $\mathbb{I} = \text{const}$

$\bar{H}_G = \mathbb{I}\,\bar{\omega}$ ($\bar{\omega}$ & $\bar{H}$ are functions of time)

suppose M_G (moment about CM) is known

$$\underbrace{\bar{M}_G = \frac{d\bar{H}_G}{dt}}_{\text{inertial reference}} \qquad \bar{M}_G = \underbrace{\left(\frac{d\bar{H}_G}{dt}\right)_{rel}}_{\text{relative reference}} + \bar{\omega} \times \bar{H}_G$$

in a symmetric body $\mathbb{I} = \begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$

$$\bar{H}_G = I_{xx}\omega_x \hat{\imath} + I_{yy}\omega_y \hat{\jmath} + I_{zz}\omega_z \hat{k}$$

$$\bar{M}_G = M_x \hat{\imath} + M_y \hat{\jmath} + M_z \hat{k}$$

$$\bar{\omega} \times \bar{H}_G = \underbrace{\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \omega_x & \omega_y & \omega_z \\ I_{xx}\omega_x & I_{yy}\omega_y & I_{zz}\omega_z \end{vmatrix}}_{\text{determinant}} = \left(I_{zz}\omega_z\omega_y - I_{yy}\omega_y\omega_z\right)\hat{\imath} - \left(I_{zz}\omega_z\omega_x - I_{xx}\omega_x\omega_z\right)\hat{\jmath} + \left(I_{yy}\omega_y\omega_x - I_{xx}\omega_x\omega_y\right)\hat{k}$$

$$\left(\frac{dH_G}{dt}\right)_{rel} = I_{xy}\dot{\omega}_x\,\hat{\imath} + I_{yy}\dot{\omega}_y\,\hat{\jmath} + I_{zz}\dot{\omega}_z\,\hat{k}$$

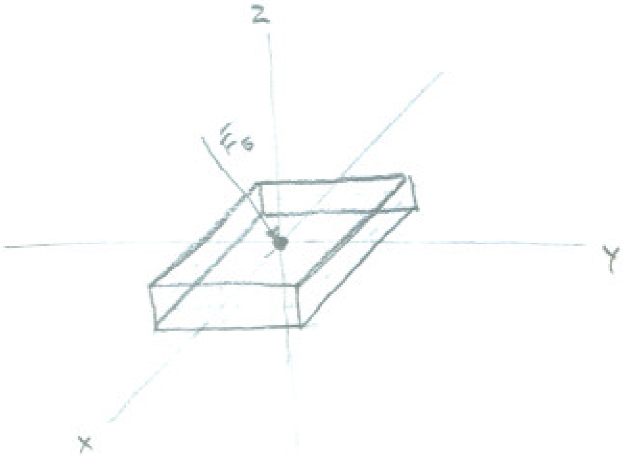
sub back into $M_G = \left(\frac{d\bar{H}_G}{dt}\right)_{rel} + \bar{\omega} \times \bar{H}_G$

$$\left.\begin{aligned} \hat{\imath}:&\quad M_x = I_{xx}\dot{\omega}_x - (I_{yy} - I_{zz})\omega_y\omega_z \\ \hat{\jmath}:&\quad M_y = I_{yy}\dot{\omega}_y - (I_{zz} - I_{xx})\omega_z\omega_x \\ \hat{k}:&\quad M_z = I_{zz}\dot{\omega}_z - (I_{xx} - I_{yy})\omega_x\omega_y \end{aligned}\right\} \text{ non-linear Eulers Equations}$$

note: in 2-D $\bar{\omega}$ & $\bar{H}$ are colinear

in 3-D $\bar{\omega}$ & $\bar{H}$ are not necessarily colinear

Book Problem :



$I_z > I_y > I_x$

torque free motion $\bar{M} = 0$

$$\dot{\omega}_x = \underbrace{\left(\frac{I_y - I_z}{I_x}\right)}_{(A)} \omega_y \omega_z$$

$$\dot{\omega}_y = \underbrace{\left(\frac{I_z - I_x}{I_y}\right)}_{(B)} \omega_x \omega_z$$

$$\dot{\omega}_z = \underbrace{\left(\frac{I_x - I_y}{I_z}\right)}_{(C)} \omega_x \omega_y$$

A is negitive
B is positive
C is negitive

rotation about z-axis

initial conditions

$\omega_z = \Omega$

$\left.\begin{array}{l}\omega_x = \epsilon \\ \omega_y = \delta\end{array}\right\}$ small errors

$\dot{\omega}_x = A \Omega \delta$

$\dot{\omega}_y = B \cdot \Omega \cdot \delta$

$\dot{\omega}_z = C \cdot \epsilon \delta \sim 0$

(drop terms of order ϵ^2, δ^2, $\epsilon\delta$)

$\dot{\epsilon} = (A\Omega)\delta$ $\qquad \ddot{\epsilon} = (A\Omega)\dot{\delta} = AB\Omega^2 \epsilon$

$\boxed{\dot{\delta} = (B\Omega)\epsilon}$ $\qquad \ddot{\delta} = (B\Omega)\dot{\epsilon} = AB\Omega^2 \delta$

$\ddot{\epsilon} - (\overbrace{AB\Omega^2}^{\text{negitive value}})\epsilon = 0$

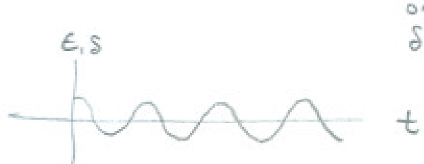
$\ddot{\epsilon} + k^2 \epsilon = 0 \qquad \epsilon = D \cdot \cos kt + E \cdot \sin kt$

or

$\delta =$ "

stable for this case :

- initially valid solution



ε,δ doesn't grow

rotation about x-axis

$$\omega_z = \epsilon \qquad \dot{\omega}_x = A\delta\Omega$$
$$\omega_y = \delta \qquad \dot{\omega}_y = B\epsilon\Omega$$
$$\omega_x = \Omega \qquad \dot{\omega}_z = \epsilon\delta\Omega \sim 0$$

$$\dot{\delta} = B\epsilon\Omega \qquad \ddot{\delta} = B\Omega\dot{\epsilon} = AB\Omega^2\delta \qquad \ddot{\delta} - AB\Omega^2\delta = 0$$
$$\dot{\epsilon} = A\delta\Omega \qquad \ddot{\epsilon} = A\Omega\dot{\delta} = AB\Omega^2\epsilon \qquad \ddot{\delta} + k^2\delta = 0$$

$$\delta = D\cdot\cos kt + E\cdot\sin kt$$
$$\epsilon = \text{''}$$

-similar stabl condition such as in z-axis rotation

rotation about y-axis

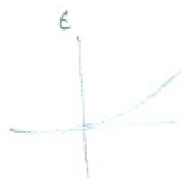
$$\omega_x = \epsilon \qquad \dot{\omega}_x = A\delta\Omega$$
$$\omega_y = \Omega \qquad \dot{\omega}_y = B\epsilon\delta \sim 0$$
$$\omega_z = \delta \qquad \dot{\omega}_z = C\epsilon\Omega$$

$$\dot{\epsilon} = A\delta\Omega \qquad \ddot{\epsilon} = A\Omega\dot{\delta} = AC\Omega^2\epsilon \qquad \ddot{\epsilon} - \overbrace{AC\Omega^2}^{\text{positive}}\epsilon = 0$$
$$\dot{\delta} = C\epsilon\Omega$$

$$\ddot{\epsilon} - k^2\epsilon = 0$$

$$\epsilon = D\cdot e^{kt} + E\cdot e^{-kt}$$

initially ϵ, δ have an exponential increase, decrease



to solve for full motion in time the non-linear Eulers equations need to be solve simultaneously, probably numerically.